

A Free-Field Lagrangian for a Gauge Theory of the CPT Symmetry

Kurt Koltko
localcpt@yahoo.com

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Abstract

A simplified mathematical approach is presented and used to find a suitable free-field Lagrangian to complete previous work on constructing a gauge theory of CPT transformations.

INTRODUCTION

Aside from curiosity, why consider gauging the CPT symmetry? We assert that gauging the CPT symmetry is necessary to further our understanding of gravitational physics. The fact that the metric spin connection formulation of general relativity can be derived from gauging the global, proper, continuous Lorentz transformations (Λ) should be sufficient motivation to consider gauging CPT. This is because PT is also a proper Lorentz transformation, and it would seem logical to include PT in gauging the *entire* proper Lorentz group of transformations, $PT\Lambda$. It is necessary to include the C operation because PT is not a universal symmetry, whereas CPT *is* a universal symmetry. In effect, we are gauging the $CPT\Lambda$ transformation of the Dirac field to induce the gauging of the full group of proper spacetime Lorentz transformations.

Another reason to gauge the CPT symmetry is because some gravitational issues can be interpreted as requiring an additional force, and the gauging of experimentally verified, global symmetries has been successful in determining the known forces. Given the historical success from gauging symmetries, we do not have much (if any) choice if one is to continue down that path because CPT is a universal, experimentally verified symmetry that does not require any extra dimensions.

For example, we argue that gauge CPT is a logical alternative to the galactic dark matter hypothesis used in the context of spiral galaxies [1, 2]. The dark matter is invoked to explain galactic rotation curves for two reasons, really. The primary reason is that the gravity produced by the observed galactic matter does not account for the motion of material (stars, HI gas) as the distance from

the galactic center increases. However, another reason is buried within - a mass independent acceleration. This acceleration is what steers a hypothetical explanation towards dark matter rather than a missing force - no forces other than gravity produce a mass independent acceleration. Thus, if gravity is to explain this strange motion, then there must be missing, unseen matter. However, another possible explanation could logically exist in the form of a missing force or an extension of general relativity because, after all, accelerations are produced by forces.

The traits of universality and of containing a proper Lorentz transformation that CPT share with Λ are the heuristic motivation that a mass independent acceleration (i.e., the equivalence principle is obeyed) would occur if any new forces or extensions of general relativity were to be unveiled by gauging $CPT\Lambda$. Indeed, the new gauge field X_μ introduced to accommodate local $CPT\Lambda$ transformations is required to contain terms of the form $x_{\mu ab}\sigma^{ab}$ ($\sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]$) which appear along with the metric spin connection containing term, $\omega_{\mu ab}\sigma^{ab}$, of general relativity.¹ Hence, the spinor fields ψ describing matter will interpret $x_{\mu ab}\sigma^{ab}$ as a gravitational effect. More detailed arguments regarding gauge CPT as an alternative to the dark matter hypothesis can be found in [1, 2].

Furthermore, gauging CPT should be of interest in the broader problem of reconciling general relativity with quantum theory. That the CPT symmetry is born neither from special relativity nor quantum theory alone, but rather the phenomenologically successful union of the two, warrants paying closer attention to CPT. Because general relativity can be obtained by making special relativity (i.e. Λ) local, it would seem interesting to include CPT when making Λ local in order to incorporate quantum theory at a fundamental level. This is analogous to expanding the early nonrenormalizable weak interaction theories by the renormalizable $SU(2) \times U(1)$ electroweak theory - any approach to quantum gravity would need to include the missing spacetime dynamical degrees of freedom unveiled by gauging CPT.

THE GAUGE THEORY OF CPT TRANSFORMATIONS

Gauging $CPT\Lambda$ is a straightforward process analogous to gauging other symmetries. The original approach ([1, 2]) used variational principles in conjunction with the action integral primarily because of the appearance of Dirac delta functionals when making the discrete, global CPT symmetry a local symmetry transformation. Dirac delta functionals only have meaning when appearing in a definite integral, and the action integral is the only obvious, natural arena for the Dirac delta functionals which occur. Also, we view the action integral to be *the* object of fundamental importance when dealing with symmetries. For a more in-depth discussion of motivation, experimental possibilities, and derivation of the mathematics used to handle these discrete transformations, see [1, 2].

¹Greek indices such as μ denote manifold coordinates. Latin indices such as a denote the local inertial coordinates. Otherwise, we use the Bjorken-Drell conventions except for σ^{ab} .

We begin with the global $CPT\Lambda$ transformation applied to the Dirac spinor ψ and the spacetime vierbein e_a^μ : $\psi \rightarrow i\gamma^5 \Lambda_\psi \psi$, and $e_a^\mu \rightarrow -e_b^\mu \Lambda_a^b$, where Λ_ψ and Λ_a^b are the spinor and spacetime representations of Λ . The CPT transformation appears as the $i\gamma^5$ in the spinor transformation and as the factor of -1 (due to PT) appearing in the vierbein transformation. We assume there is no non-trivial *spacetime* analog of the C operation, i.e., there is no such thing as "anti-spacetime" (an attempt to find a non-trivial spacetime C operation is contained in [2]). There are two steps in making these transformations local. First, we make Λ a local transformation. Then, we introduce a real, differentiable function f to be used as the argument of unit step functions Θ , where we define $\Theta[f \leq 0] = 0$, and $\Theta[f > 0] = 1$. In *arbitrarily chosen* regions where we want to perform the $CPT\Lambda$ operation, we set $f > 0$. In the regions we leave alone, we set $f < 0$. The boundaries between the regions where $CPT\Lambda$ is applied and where it is not are given by $f = 0$. It is to be emphasized that f is *not* a physical field, just a parameter used to make CPT local; therefore, the function f must disappear from the Lagrangian and ensuing field equations. One thus obtains the local $CPT\Lambda$ transformations [1]:

$$\begin{aligned} e_a^\mu &\rightarrow \Theta[-f] e_a^\mu - \Theta[f] e_b^\mu \Lambda_a^b, \\ \psi &\rightarrow \Theta[-f] \psi + \Theta[f] i\gamma^5 \Lambda_\psi \psi \equiv U\psi, \text{ where} \\ U &= \Theta[-f] I + \Theta[f] i\gamma^5 \Lambda_\psi. \end{aligned}$$

We denote the local $CPT\Lambda$ transformation by the customary U even though the transformation is not unitary. Unitarity is not necessary in making gauge theories. For example, Λ is not unitary and is used to derive the metric spin connection formulation of general relativity as a gauge theory. We also note that the presence or absence of a conservation law associated with a global symmetry transformation has no relevance to constructing the ensuing gauge theory. Again, returning to Λ , we note that there are no conservation laws associated with global boosts.

To proceed further, we must take derivatives and make products from the above transformations. Because of the discrete nature of the transformations, various discontinuities naturally arise. The action integral, variational methods, and elementary functional analysis lead to the following important rules needed to handle the discontinuities ($[1, 2]$): $\Theta[-f] \Theta[f] = 0$, $(\Theta[\pm f])^n = \Theta[\pm f]$ (n is a positive integer ≥ 1), and $\partial_\mu \Theta[\pm f] = \pm \delta[f] \partial_\mu f$, where $\delta[f]$ is a Dirac delta functional. Application of these rules to the above transformations immediately gives us:

$$\begin{aligned} U^{-1} &= \Theta[-f] I - \Theta[f] i\gamma^5 \Lambda_\psi, \\ \partial_\mu U &= \Theta[f] i\gamma^5 \partial_\mu \Lambda_\psi + \delta[f] \partial_\mu f (i\gamma^5 \Lambda_\psi - I), \text{ and} \\ (\partial_\mu U) U^{-1} &= \Theta[f] (\partial_\mu \Lambda_\psi) \Lambda_\psi + \Theta[-f] \delta[f] \partial_\mu f (i\gamma^5 \Lambda_\psi - I) \\ &\quad + \Theta[f] \delta[f] \partial_\mu f (I + i\gamma^5 \Lambda_\psi), \end{aligned}$$

where $\Lambda_{\bar{\psi}}$ is the inverse of Λ_{ψ} (also, $\bar{\psi} \rightarrow i\bar{\psi}\gamma^5\Lambda_{\bar{\psi}}$ under global $CPT\Lambda$).

Everything follows from these equations. The Dirac equation is obviously not invariant under U , so we introduce a minimally coupled gauge field as part of a covariant derivative acting on ψ . As always, a gauge field Z_{μ} associated with U must transform as $Z_{\mu} \rightarrow UZ_{\mu}U^{-1} - \frac{1}{\beta}(\partial_{\mu}U)U^{-1}$, where β is the coupling constant. Because $(\partial_{\mu}U)U^{-1}$ has terms containing I and γ^5 in addition to σ^{ab} , we immediately see from the linear independence of I , γ^5 , and σ^{ab} that the metric spin connection term, $\omega_{\mu ab}\sigma^{ab}$, cannot compensate for local $CPT\Lambda$ transformations (see [1] for an alternative proof). Therefore, we introduce a new gauge field X_{μ} of the form $X_{\mu} = x_{\mu I}I + x_{\mu 5}\gamma^5 + x_{\mu ab}\sigma^{ab}$ as part of a covariant derivative $D_{\mu} = \partial_{\mu} + \frac{1}{2}\omega_{\mu ab}\sigma^{ab} + \beta X_{\mu}$. In other words, $Z_{\mu} = \frac{1}{2\beta}\omega_{\mu ab}\sigma^{ab} + X_{\mu}$. One can easily verify that the following transformation (see [1] for the derivation) for X_{μ} gives $D_{\mu} \rightarrow UD_{\mu}U^{-1}$:

$$\begin{aligned} X_{\mu} &\rightarrow \Theta[-f]X_{\mu} + \Theta[f]\Lambda_{\psi}X_{\mu}\Lambda_{\bar{\psi}} + \Theta[-f]\delta[f]Y_{\mu} + \Theta[f]\delta[f]\tilde{Y}_{\mu}, \text{ where} \\ Y_{\mu} &= \beta^{-1}\left[\partial_{\mu}f(I - i\gamma^5\Lambda_{\psi}) - \frac{1}{2}\varsigma_{\mu ab}\sigma^{ab}\right], \\ \tilde{Y}_{\mu} &= \beta^{-1}\left[\partial_{\mu}f(-I - i\gamma^5\Lambda_{\bar{\psi}}) - \frac{1}{2}\tilde{\varsigma}_{\mu ab}\sigma^{ab}\right]. \end{aligned}$$

The terms $\varsigma_{\mu ab}$, $\tilde{\varsigma}_{\mu ab}$ come from the differentiation of the transformed vierbein contained in the metric spin connection, $\omega_{\mu ab}$, under local $CPT\Lambda$:

$$\omega_{\mu ab} \rightarrow \Theta[-f]\omega_{\mu ab} + \Theta[f]\tilde{\omega}_{\mu ab} + \Theta[-f]\delta[f]\varsigma_{\mu ab} + \Theta[f]\delta[f]\tilde{\varsigma}_{\mu ab},$$

where $\tilde{\omega}_{\mu ab}$ satisfies $\tilde{\omega}_{\mu ab}\sigma^{ab} = \Lambda_{\psi}\omega_{\mu ab}\sigma^{ab}\Lambda_{\bar{\psi}} - 2(\partial_{\mu}\Lambda_{\psi})\Lambda_{\bar{\psi}}$. Explicit expressions [1] for $\omega_{\mu ab}$, $\tilde{\omega}_{\mu ab}$, $\varsigma_{\mu ab}$, and $\tilde{\varsigma}_{\mu ab}$ are in the appendix.

We are now ready to find the Lagrangian using the usual machinery of gauge theories. First, we know that X_{μ} must be massless (see [1] for an alternative proof). We also know that $[D_{\mu}, D_{\nu}]$ transforms gauge covariantly. Unfortunately, using $Tr\left\{[D_{\mu}, D_{\nu}][D^{\mu}, D^{\nu}]^{\dagger}\right\}$ as a free-field term introduces R^2 into the Lagrangian instead of the required R of general relativity, where R is the Einstein-Hilbert scalar curvature term formed from just the metric spin connection and a couple of vierbeins. Use of just $[D_{\mu}, D_{\nu}]$ is problematic because it is not clear what to contract it with, courtesy of the absence of the Latin (i.e. inertial) indices in the $x_{\mu I}$ and $x_{\mu 5}$ terms. Also, at some point in the total Lagrangian, the $\omega_{\mu ab}$ and $x_{\mu ab}$ terms must appear on an unequal footing outside of the $\frac{1}{2}\omega_{\mu ab}\sigma^{ab} + \beta x_{\mu ab}\sigma^{ab}$ combination - otherwise, the sole purpose of $x_{\mu ab}$ is just to be a "fudge factor" introduced to get rid of the $\varsigma_{\mu ab}$, $\tilde{\varsigma}_{\mu ab}$ terms appearing in the transformation of $\omega_{\mu ab}$.

The key to constructing a viable Lagrangian is that R is invariant and covariant under local $CPT\Lambda$ transformations even though $\omega_{\mu ab}$ is not [1]. (Briefly, the R formed from the metric spin connection, $\omega_{\mu ab}$, is the same as the familiar R formed from the metric tensor $g^{\mu\nu}$, and the R from $g^{\mu\nu}$ is both gauge invariant and gauge covariant. Gauge invariance follows because $g^{\mu\nu} = \eta^{ab}e_a^{\mu}e_b^{\nu} \rightarrow$

$\Theta [-f] g^{\mu\nu} + \Theta [f] g^{\mu\nu}$, where η^{ab} is the Minkowski metric tensor. From this we immediately see that derivatives of the metric tensor also transform in the same manner, $\partial_\rho g^{\mu\nu} \rightarrow \Theta [-f] \partial_\rho g^{\mu\nu} + \Theta [f] \partial_\rho g^{\mu\nu}$. So, the metric tensor, R , and any functions of these only pick up removable singularities under local $CPT\Lambda$ transformations. These singularities have no effect on the action integral and are therefore dropped. Again, the action integral and variational approach ([1, 2]) are of fundamental importance in dealing with singularities. The proof that R is gauge covariant is in the appendix.) Expanding $[D_\mu, D_\nu]$ gives us:

$$\begin{aligned} [D_\mu, D_\nu] &= \frac{\beta}{2} (\omega_{\mu ab} [\sigma^{ab}, X_\nu] - \omega_{\nu ab} [\sigma^{ab}, X_\mu]) \\ &\quad + \beta^2 [X_\mu, X_\nu] + \beta (\partial_\mu X_\nu - \partial_\nu X_\mu) \\ &\quad + \left\{ \frac{1}{4} \omega_{\mu ab} \omega_{\nu cd} [\sigma^{ab}, \sigma^{cd}] + \frac{1}{2} (\partial_\mu \omega_{\nu ab} - \partial_\nu \omega_{\mu ab}) \sigma^{ab} \right\}. \end{aligned}$$

The terms in $\{\dots\}$ give R upon contraction with a couple of vierbein. Because both $[D_\mu, D_\nu]$ and the $\{\dots\}$ terms transform gauge covariantly, we see that the remaining terms - denoted as $H_{\mu\nu}$ - must also transform gauge covariantly. We can now form a gauge covariant Lagrangian term from $H_{\mu\nu}$ as $Tr \{H_{\mu\nu} H^{\mu\nu\dagger}\}$. The Hermitian action we obtain from minimal coupling of $\omega_{\mu ab}$, X_μ , and ψ is thus [1]:

$$\begin{aligned} S &= \int \{ \kappa R - m \bar{\psi} \psi + \frac{i}{2} e_a^\mu \bar{\psi} \gamma^a \left(\partial_\mu \psi + \frac{1}{2} \omega_{\mu bc} \sigma^{bc} \psi + \beta X_\mu \psi \right) \} e d^4 x \\ &\quad - \int \left\{ \frac{i}{2} e_a^\mu \left(\partial_\mu \bar{\psi} - \frac{1}{2} \omega_{\mu bc} \bar{\psi} \sigma^{bc} + \beta \bar{\psi} \gamma^0 X_\mu^\dagger \gamma^0 \right) \gamma^a \psi \right\} e d^4 x \\ &\quad + \int \left\{ \frac{1}{4} Tr (H_{\mu\nu} H^{\mu\nu\dagger}) \right\} e d^4 x, \text{ where} \end{aligned}$$

$e = \det(e_a^\mu)$ and $H_{\mu\nu} = \frac{\beta}{2} (\omega_{\mu ab} [\sigma^{ab}, X_\nu] - \omega_{\nu ab} [\sigma^{ab}, X_\mu]) + \beta^2 [X_\mu, X_\nu] + \beta (\partial_\mu X_\nu - \partial_\nu X_\mu)$. The resulting equations of motion are [1] (*h.c.* means Hermitian conjugate of the preceding term):

$$i e_a^\mu \gamma^a \left(\partial_\mu \psi + \frac{1}{2} \omega_{\mu bc} \sigma^{bc} \psi + \beta X_\mu \psi \right) - m \psi = 0 \text{ (the Dirac equation),}$$

$$\begin{aligned} i \bar{\psi} e_a^\mu \gamma^a \psi &= 4\beta (\partial^\mu x_I^\nu - \partial^\nu x_I^\mu)_{;\nu} \text{ and} \\ i \bar{\psi} e_a^\mu \gamma^a \gamma^5 \psi &= 4\beta (\partial^\mu x_5^\nu - \partial^\nu x_5^\mu)_{;\nu} \text{ (the "chiral" terms of } X_\mu), \end{aligned}$$

$$\begin{aligned}
i\bar{\psi}\sigma^{jk}e_a^\mu\gamma^a\psi &= \left\{Tr\left[\left(\frac{\beta}{2}(\omega^{\nu ab}[\sigma_{ab}, X^\mu] - \omega^{\mu ab}[\sigma_{ab}, X^\nu]) + \beta^2[X^\nu, X^\mu]\right.\right.\right. \\
&\quad \left.\left.+\beta(\partial^\nu X^\mu - \partial^\mu X^\nu)\sigma^{jk\dagger}\right]\right\}_{;\nu} \\
&\quad -Tr\left[\left(\frac{\beta}{2}(\omega^{\nu ab}[\sigma_{ab}, X^\mu] - \omega^{\mu ab}[\sigma_{ab}, X^\nu]) + \beta^2[X^\nu, X^\mu]\right.\right. \\
&\quad \left.\left.+\beta(\partial^\nu X^\mu - \partial^\mu X^\nu)\left(\frac{1}{2}\omega_{\nu cd} + \beta x_{\nu cd}^*\right)\right.\right. \\
&\quad \left.\left.\times\gamma^0[\sigma^{dc}, \sigma^{jk}]\gamma^0\right]\right\} \text{ (the } x^{\mu jk} \text{ terms of } X^\mu),
\end{aligned}$$

$$\begin{aligned}
(e^{q\lambda}e^{p\alpha} - e^{q\alpha}e^{p\lambda})_{;\alpha} &= \omega_\mu^{pn}(e_n^\lambda e^{q\mu} - e^{q\lambda}e_n^\mu) + \omega_\mu^{nq}(e_n^\lambda e^{p\mu} - e_n^\mu e^{p\lambda}) \\
&\quad + \frac{i}{4}\bar{\psi}\{\gamma^\lambda, \sigma^{qp}\}\psi \\
&\quad + \frac{1}{4}Tr\left\{\left[\beta[\sigma^{qp}, X^\mu]\left(\frac{\beta}{2}(\omega_{ab}^\lambda[X_\mu^\dagger, \sigma^{ab\dagger}] \right.\right.\right. \\
&\quad \left.\left.\left.-\omega_{\mu ab}[X^{\lambda\dagger}, \sigma^{ab\dagger}]) + \beta^2[X_\mu^\dagger, X^{\lambda\dagger}]\right.\right.\right. \\
&\quad \left.\left.+\beta g^{\lambda\nu}(\partial_\nu X_\mu^\dagger - \partial_\mu X_\nu^\dagger)\right]\right\} + h.c. \\
&\quad \text{(Palatini spin connection), and}
\end{aligned}$$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\kappa^2 T_{\mu\nu} \text{ (general relativity).}$$

We include the Palatini spin connection for completeness. The metric spin connection can be obtained from the above Palatini spin connection equations by setting $X_\mu = \psi = 0$.

The $x_{\mu ab}$ terms are the obvious terms to examine as possible explanations for current gravitational issues because they couple to the matter fields in the same manner as the spin connection. Experimental speculations and crude predictions can be found in [1, 2]. Also, the $x_{\mu ab}$ field equations should explain the Baryonic Tully-Fisher law and the Faber-Jackson law; however, work on this is still in progress. The physical implications of the "chiral" terms, $x_{\mu I}$ and $x_{\mu 5}$, and their associated anomalies are not developed enough to warrant a discussion.

APPENDIX

We show that the Einstein-Hilbert R transforms gauge covariantly. Or, equivalently, we need to show that $R_{\mu\nu ab}\sigma^{ab} \rightarrow UR_{\mu\nu ab}\sigma^{ab}U^{-1}$ under local $CPTA$ transformations (the transformation of the vierbeins used in the contraction of $R_{\mu\nu ab}$ to obtain R cause no complications because they do not introduce any delta functionals). We begin with re-expressing $R_{\mu\nu ab}$ in terms of the more familiar form of the Riemann curvature tensor: $R_{\mu\nu ab} = R_{\mu\nu\rho\sigma}e_a^\rho e_b^\sigma$. We recall that $R_{\mu\nu\rho\sigma}$ is invariant under local $CPTA$ transformations because $R_{\mu\nu\rho\sigma}$ is comprised of the metric tensor and its derivatives. So, we have:

$$\begin{aligned}
R_{\mu\nu ab}\sigma^{ab} &= R_{\mu\nu\rho\sigma}e_a^\rho e_b^\sigma\sigma^{ab} \\
&\rightarrow R_{\mu\nu\rho\sigma}(\Theta[-f]e_a^\rho - \Theta[f]e_c^\rho\Lambda_a^c)(\Theta[-f]e_b^\sigma - \Theta[f]e_d^\sigma\Lambda_b^d)\sigma^{ab} \\
&= R_{\mu\nu\rho\sigma}(\Theta[-f]e_a^\rho e_b^\sigma + \Theta[f]\Lambda_a^c\Lambda_b^d e_c^\rho e_d^\sigma)\sigma^{ab}.
\end{aligned}$$

We compare this with $UR_{\mu\nu ab}\sigma^{ab}U^{-1}$:

$$\begin{aligned}
UR_{\mu\nu ab}\sigma^{ab}U^{-1} &= (\Theta[-f]I + \Theta[f]i\gamma^5\Lambda_\psi)R_{\mu\nu ab}\sigma^{ab}(\Theta[-f]I - \Theta[f]i\gamma^5\Lambda_\psi^-) \\
&= (\Theta[-f]R_{\mu\nu ab}\sigma^{ab} + \Theta[f]iR_{\mu\nu ab}\gamma^5\Lambda_\psi\sigma^{ab})(\Theta[-f]I - \Theta[f]i\gamma^5\Lambda_\psi^-) \\
&= \Theta[-f]R_{\mu\nu ab}\sigma^{ab} + \Theta[f]R_{\mu\nu ab}\gamma^5\Lambda_\psi\sigma^{ab}\gamma^5\Lambda_\psi^- \\
&= \Theta[-f]R_{\mu\nu ab}\sigma^{ab} + \Theta[f]R_{\mu\nu ab}\Lambda_\psi\sigma^{ab}\Lambda_\psi^- \\
&= \Theta[-f]R_{\mu\nu ab}\sigma^{ab} + \Theta[f]R_{\mu\nu ab}\Lambda_c^a\Lambda_d^b\sigma^{cd} \\
&= \Theta[-f]R_{\mu\nu\rho\sigma}e_a^\rho e_b^\sigma\sigma^{ab} + \Theta[f]R_{\mu\nu\rho\sigma}e_a^\rho e_b^\sigma\Lambda_c^a\Lambda_d^b\sigma^{cd} \\
&= R_{\mu\nu\rho\sigma}(\Theta[-f]e_a^\rho e_b^\sigma + \Theta[f]\Lambda_a^c\Lambda_b^d e_c^\rho e_d^\sigma)\sigma^{ab}.
\end{aligned}$$

QED

The expressions for $\omega_{\mu ab}$, $\tilde{\omega}_{\mu ab}$, $\varsigma_{\mu ab}$, and $\tilde{\varsigma}_{\mu ab}$ are:

$$\begin{aligned}
\omega_{\mu ab} &= \frac{1}{2}e_a^\nu(\partial_\mu e_{b\nu} - \partial_\nu e_{b\mu}) - \frac{1}{2}e_b^\nu(\partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}) \\
&\quad - \frac{1}{2}e_a^\rho e_b^\sigma(\partial_\rho e_{r\sigma} - \partial_\sigma e_{r\rho})e_\mu^r. \\
\tilde{\omega}_{\mu ab} &= \frac{1}{2}\{e_d^\nu(\partial_\mu e_{c\nu} - \partial_\nu e_{c\mu})(\Lambda_a^d\Lambda_b^c - \Lambda_b^d\Lambda_a^c) \\
&\quad + \eta_{cd}(\Lambda_a^d\partial_\mu\Lambda_b^c - \Lambda_b^d\partial_\mu\Lambda_a^c) - e_d^\nu e_{c\mu}(\Lambda_a^d\partial_\nu\Lambda_b^c - \Lambda_b^d\partial_\nu\Lambda_a^c) \\
&\quad - \Lambda_a^c\Lambda_b^r e_c^\rho e_r^\sigma(\partial_\rho e_{p\sigma} - \partial_\sigma e_{p\rho})e_\mu^p \\
&\quad - \Lambda_a^c\Lambda_b^r(\partial_\rho\Lambda_p^s)\Lambda_d^p(\eta_{rs}e_c^\rho - \eta_{cs}e_r^\rho)e_\mu^d\}. \\
\tilde{\varsigma}_{\mu ab} &= \frac{1}{2}\{\partial_\nu f[e_d^\nu(\Lambda_b^d e_{a\mu} - \Lambda_a^d e_{b\mu}) - e_d^\nu e_{c\mu}(\Lambda_a^d\Lambda_b^c - \Lambda_b^d\Lambda_a^c) \\
&\quad - e_c^\nu e_\mu^d\Lambda_d^p(\eta_{pr}(\Lambda_a^c\Lambda_b^r - \Lambda_a^r\Lambda_b^c) + (\eta_{bp}\Lambda_a^c - \eta_{ap}\Lambda_b^c)))] \\
&\quad + \partial_\mu f(\eta_{bd}\Lambda_a^d - \eta_{ad}\Lambda_b^d)\}. \\
\varsigma_{\mu ab} &= \frac{1}{2}\{\partial_\nu f[2(e_a^\nu e_{b\mu} - e_b^\nu e_{a\mu}) + e_{c\mu}(e_a^\nu\Lambda_b^c - e_b^\nu\Lambda_a^c) \\
&\quad - e_\mu^d\Lambda_d^c(\eta_{ac}e_b^\nu - \eta_{bc}e_a^\nu)] - \partial_\mu f(\eta_{ac}\Lambda_b^c - \eta_{bc}\Lambda_a^c)\}.
\end{aligned}$$

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